

# THE ABELIAN MONOID OF FUSION-STABLE FINITE SETS IS FREE

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**ABSTRACT.** For a finite group  $G$  with a Sylow  $p$ -subgroup  $S$ , we say that a finite set with an action of  $S$  is  $G$ -stable if the action is unchanged up to isomorphism when we act through conjugation maps in  $G$ . We show that the abelian monoid of isomorphism classes of  $G$ -stable  $S$ -sets is free, and we give an explicit construction of the basis, whose elements are in one-to-one correspondence with  $G$ -conjugacy classes of subgroups in  $S$ . As a main tool for proving freeness, we describe the Burnside ring of a saturated fusion system, and its embedding into a suitable associated ghost ring.

## 1. INTRODUCTION

Given a finite group  $G$  acting on a finite set  $X$ , we can restrict the action to a Sylow  $p$ -subgroup  $S$  of  $G$ . The resulting  $S$ -set has the property that it stays the same (up to  $S$ -isomorphism) whenever we change the action via a conjugation map from  $G$ . More precisely, if  $P \leq S$  is a subgroup and  $\varphi: P \rightarrow S$  is a homomorphism given by conjugation with some element of  $G$ , we can turn  $X$  into a  $P$ -set by using  $\varphi$  to define the action  $p.x := \varphi(p)x$ . We denote the resulting  $P$ -set by  ${}_{P,\varphi}X$ . In particular when  $\text{incl}: P \rightarrow S$  is the inclusion map,  ${}_{P,\text{incl}}X$  has the usual restriction of the  $S$ -action to  $P$ . When a finite  $S$ -set  $X$  is the restriction of a  $G$ -set, then  $X$  has the property

$$(1.1) \quad {}_{P,\varphi}X \text{ is isomorphic to } {}_{P,\text{incl}}X \text{ as } P\text{-sets, for all } P \leq S \text{ and homomorphisms } \varphi: P \rightarrow S \text{ induced by } G\text{-conjugation.}$$

Any  $S$ -set with property (1.1) is called  $G$ -stable. Whenever we restrict a  $G$ -set to  $S$ , the resulting  $S$ -set is  $G$ -stable; however there are  $G$ -stable  $S$ -sets whose  $S$ -actions do not extend to actions of  $G$ .

The isomorphism classes of finite  $S$ -sets form a semiring with disjoint union as addition and cartesian product as multiplication. The collection of  $G$ -stable  $S$ -sets is closed under addition and multiplication, hence  $G$ -stable sets form a subsemiring.

**Theorem A** (for finite groups). *Let  $G$  be a finite group with Sylow  $p$ -group  $S$ .*

*Every  $G$ -stable  $S$ -set splits uniquely (up to  $S$ -isomorphism) as a disjoint union of irreducible  $G$ -stable sets, and there is a one-to-one correspondence between the irreducible  $G$ -stable sets and  $G$ -conjugacy classes of subgroups in  $S$ .*

*Hence the semiring of  $G$ -stable sets is additively a free commutative monoid with rank equal to the number of  $G$ -conjugacy classes of subgroups in  $S$ .*

As part of the proof we give an explicit construction of the irreducible  $G$ -stable sets.

It is a well-known fact that any finite  $S$ -set splits uniquely into orbits/transitive  $S$ -sets; and the isomorphism type of a transitive set  $S/P$  depends only on the subgroup  $P$  up to  $S$ -conjugation. Theorem A then states that this fact generalizes nicely to  $G$ -stable  $S$ -sets, which is less obvious than it might first appear.

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If we consider  $G$ -sets and restrict their actions to  $S$ , then two non-isomorphic  $G$ -sets might very well become isomorphic as  $S$ -sets. Therefore even though finite  $G$ -sets decompose uniquely into orbits, we have no guarantee that this decomposition remains unique when we restrict the actions to the Sylow subgroup  $S$ . In fact, uniqueness of decompositions fails in general when we consider restrictions of  $G$ -sets to  $S$ , as demonstrated in example 4.3 for the symmetric group  $S_5$  and its Sylow 2-subgroup.

It then comes as a surprise that if we consider *all*  $G$ -stable  $S$ -sets, and not just the restrictions of actual  $G$ -sets, we can once more write stable sets as a disjoint union of irreducibles in a unique way.

The proof of theorem A relies only on the way  $G$  acts on the subgroups of  $S$  by conjugation. We therefore state and prove the theorem in general for abstract saturated fusion systems, which model the action of a group on a Sylow subgroup.

If  $\mathcal{F}$  is a fusion system over a  $p$ -group  $S$ , we say that an  $S$ -set  $X$  is  $\mathcal{F}$ -stable if it satisfies

$$(1.2) \quad \begin{array}{l} P, \varphi X \text{ is isomorphic to } P, \text{incl} X \text{ as } P\text{-sets, for all } P \leq S \text{ and homomorphisms} \\ \varphi: P \rightarrow S \text{ in } \mathcal{F}. \end{array}$$

The  $\mathcal{F}$ -stable  $S$ -sets form a semiring since the disjoint union and cartesian product of  $\mathcal{F}$ -stable sets is again  $\mathcal{F}$ -stable; and theorem A then generalizes to

**Theorem A** (for fusion systems). *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ .*

*Every  $\mathcal{F}$ -stable  $S$ -set splits uniquely (up to  $S$ -isomorphism) as a disjoint union of irreducible  $\mathcal{F}$ -stable sets, and there is a one-to-one correspondence between the irreducible  $\mathcal{F}$ -stable sets and conjugacy/isomorphism classes of subgroups in the fusion system  $\mathcal{F}$ .*

*Hence the semiring of  $\mathcal{F}$ -stable sets is additively a free commutative monoid with rank equal to the number of conjugacy classes of subgroups in  $\mathcal{F}$ .*

An important tool in proving theorem A, is the subring consisting of  $\mathcal{F}$ -stable elements inside the Burnside ring  $A(S)$  of  $S$ , where the  $\mathcal{F}$ -stable elements satisfy a property similar to (1.2). This subring will be the Grothendieck group of the semiring of  $\mathcal{F}$ -stable sets, and we call it *the Burnside ring of  $\mathcal{F}$*  denoted by  $A(\mathcal{F})$ .

By restriction, the Burnside ring of  $\mathcal{F}$  inherits the homomorphism of marks from  $A(S)$ , embedding  $A(\mathcal{F})$  into a product of a suitable number of copies of  $\mathbb{Z}$ . As a main step in proving theorem A, we show that this mark homomorphism has properties analogous the mark homomorphism for groups:

**Theorem B.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ , and let  $A(\mathcal{F})$  be the Burnside ring of  $\mathcal{F}$  – i.e. the subring consisting of the  $\mathcal{F}$ -stable elements in the Burnside ring of  $S$ .*

*Then there is a ring homomorphism  $\Phi$  and a group homomorphism  $\Psi$  which fit together in the following short-exact sequence:*

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \prod_{\substack{\text{conj. classes} \\ \text{in } \mathcal{F}}} \mathbb{Z} \xrightarrow{\Psi} \prod_{\substack{[P]_{\mathcal{F}} \text{ conj. class in } \mathcal{F}, \\ P \text{ fully } \mathcal{F}\text{-normalized}}} \mathbb{Z}/|W_S P| \rightarrow 0,$$

where  $W_S P := N_S P/P$ .

$\Phi$  comes from restricting the mark homomorphism of  $A(S)$ , and  $\Psi$  is given by the  $[P]_{\mathcal{F}}$ -coordinate functions

$$\Psi_P(f) := \sum_{\bar{s} \in W_S P} f_{\langle s \rangle P} \pmod{|W_S P|}$$

when  $P$  is a fully normalized representative of the conjugacy class  $[P]_{\mathcal{F}}$  in  $\mathcal{F}$ . Here  $\Psi_P = \Psi_{P'}$  if  $P \sim_{\mathcal{F}} P'$  are both fully normalized.

This generalizes previous results by Dress and others (see [4], [3, Section 1] or [7]) concerning the mark homomorphism and congruence relations for Burnside rings of finite groups, which also constitutes most of the proof of theorem B. Though it is easier to prove and less surprising that theorem A, we still draw attention to theorem B here because of how useful such a characterisation of Burnside rings in terms of marks can be.

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## 2. FUSION SYSTEMS

The next few pages contain a very short introduction to fusion systems. The aim is to introduce the terminology from the theory of fusion systems that will be used in the paper, and to establish the relevant notation. For a proper introduction to fusion systems see for instance Part I of "Fusion Systems in Algebra and Topology" by Aschbacher, Kessar and Oliver, [1].

**Definition 2.1.** A *fusion system*  $\mathcal{F}$  on a  $p$ -group  $S$ , is a category where the objects are the subgroups of  $S$ , and for all  $P, Q \leq S$  the morphisms must satisfy:

- (i) Every morphism  $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$  is an injective group homomorphism, and the composition of morphisms in  $\mathcal{F}$  is just composition of group homomorphisms.
- (ii)  $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q)$ , where

$$\text{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}$$

is the set of group homomorphisms  $P \rightarrow Q$  induced by  $S$ -conjugation.

- (iii) For every morphism  $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ , the group isomorphisms  $\varphi: P \rightarrow \varphi P$  and  $\varphi^{-1}: \varphi P \rightarrow P$  are elements of  $\text{Mor}_{\mathcal{F}}(P, \varphi P)$  and  $\text{Mor}_{\mathcal{F}}(\varphi P, P)$  respectively.

We also write  $\text{Hom}_{\mathcal{F}}(P, Q)$  or just  $\mathcal{F}(P, Q)$  for the morphism set  $\text{Mor}_{\mathcal{F}}(P, Q)$ ; and the group  $\mathcal{F}(P, P)$  of automorphisms is denoted by  $\text{Aut}_{\mathcal{F}}(P)$ .

The canonical example of a fusion system comes from a finite group  $G$  with a given  $p$ -subgroup  $S$ . The fusion system of  $G$  over  $S$ , denoted  $\mathcal{F}_S(G)$ , is the fusion system on  $S$  where the morphisms from  $P \leq S$  to  $Q \leq S$  are the homomorphisms induced by  $G$ -conjugation:

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) := \text{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\}.$$

A particular case is the fusion system  $\mathcal{F}_S(S)$  consisting only of the homomorphisms induced by  $S$ -conjugation.

Let  $\mathcal{F}$  be an abstract fusion system on  $S$ . We say that two subgroups  $P, Q \leq S$  are  $\mathcal{F}$ -conjugate, written  $P \sim_{\mathcal{F}} Q$ , if they are isomorphic in  $\mathcal{F}$ , i.e. there exists a group isomorphism  $\varphi \in \mathcal{F}(P, Q)$ .  $\mathcal{F}$ -conjugation is an equivalence relation, and the set of  $\mathcal{F}$ -conjugates to  $P$  is denoted by  $[P]_{\mathcal{F}}$ . The set of all  $\mathcal{F}$ -conjugacy classes of subgroups in  $S$  is denoted by  $Cl(\mathcal{F})$ . Similarly, we write  $P \sim_S Q$  if  $P$  and  $Q$  are  $S$ -conjugate, the  $S$ -conjugacy class of  $P$  is written  $[P]_S$  or just  $[P]$ , and we write  $Cl(S)$  for the set of  $S$ -conjugacy classes of subgroups in  $S$ . Since all  $S$ -conjugation maps are in  $\mathcal{F}$ , any  $\mathcal{F}$ -conjugacy class  $[P]_{\mathcal{F}}$  can be partitioned into disjoint  $S$ -conjugacy classes of subgroups  $Q \in [P]_{\mathcal{F}}$ .

We say that  $Q$  is  $\mathcal{F}$ - or  $S$ -subconjugate to  $P$  if  $Q$  is respectively  $\mathcal{F}$ - or  $S$ -conjugate to a subgroup of  $P$ , and we denote this by  $Q \lesssim_{\mathcal{F}} P$  or  $Q \lesssim_S P$  respectively. In the case where  $\mathcal{F} = \mathcal{F}_S(G)$ , we have  $Q \lesssim_{\mathcal{F}} P$  if and only if  $Q$  is  $G$ -conjugate to a subgroup of  $P$ ; and the  $\mathcal{F}$ -conjugates of  $P$ , are just those  $G$ -conjugates of  $P$  which are contained in  $S$ .

A subgroup  $P \leq S$  is said to be *fully  $\mathcal{F}$ -normalized* if  $|N_S P| \geq |N_S Q|$  for all  $Q \in [P]_{\mathcal{F}}$ ; and similarly  $P$  is *fully  $\mathcal{F}$ -centralized* if  $|C_S P| \geq |C_S Q|$  for all  $Q \in [P]_{\mathcal{F}}$ .

**Definition 2.2.** A fusion system  $\mathcal{F}$  on  $S$  is said to be *saturated* if the following properties are satisfied for all  $P \leq S$ :

- (i) If  $P$  is fully  $\mathcal{F}$ -normalized, then  $P$  is fully  $\mathcal{F}$ -centralized, and  $\text{Aut}_S(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ .
- (ii) Every homomorphism  $\varphi \in \mathcal{F}(P, S)$  where  $\varphi(P)$  is fully  $\mathcal{F}$ -centralized, extends to a homomorphism  $\varphi \in \mathcal{F}(N_{\varphi}, S)$  where

$$N_{\varphi} := \{x \in N_S(P) \mid \exists y \in S: \varphi \circ c_x = c_y \circ \varphi\}.$$

The saturated fusion systems form a class of particularly nice fusion systems, and the saturation axiom are a way to emulate the Sylow theorems for finite groups. In particular, whenever  $S$  is a Sylow  $p$ -subgroup of  $G$ , then the Sylow theorems imply that the induced fusion system  $\mathcal{F}_S(G)$  is saturated (see e.g. [1, Theorem 2.3]).

In this paper, we shall rarely use the defining properties of saturated fusion systems directly. We shall instead mainly use the following lifting property that saturated fusion systems satisfy:

**Lemma 2.3** ([6]). *Let  $\mathcal{F}$  be saturated. Suppose that  $P \leq S$  is fully normalized, then for each  $Q \in [P]_{\mathcal{F}}$  there exists a homomorphism  $\varphi \in \mathcal{F}(N_S Q, N_S P)$  with  $\varphi(Q) = P$ .*

For the proof, see lemma 4.5 of [6] or lemma 2.6(c) of [1].

### 3. BURNSIDE RINGS FOR GROUPS

In this section we consider the Burnside ring of a finite group  $S$ , and the semiring of finite  $S$ -sets. We recall the structure of the Burnside ring  $A(S)$  and how to describe the elements and operations of  $A(S)$  in terms of fixed points and the homomorphism of marks. In this section  $S$  can be any finite group, but later we shall only need the case where  $S$  is a  $p$ -group.

We consider finite  $S$ -sets up to  $S$ -isomorphism, and let  $A_+(S)$  denote the set of isomorphism classes. Given a finite  $S$ -set  $X$ , we denote the isomorphism class of  $X$  by  $[X] \in A_+(S)$ .  $A_+(S)$  is a commutative semiring with disjoint union as addition and cartesian product as multiplication, and additively  $A_+(S)$  is a free commutative monoid, where the basis consists of the (isomorphism classes) of transitive  $S$  sets, i.e.  $[S/P]$  where  $P$  is a subgroup of  $S$ . Two transitive  $S$ -sets  $S/P$  and  $S/Q$  are isomorphic if and only if  $P$  is conjugate to  $Q$  in  $S$ .

To describe the multiplication of the semiring  $A_+(S)$ , it is enough to know the products of basis elements  $[S/P]$  and  $[S/Q]$ . By taking the product  $(S/P) \times (S/Q)$  and considering how it breaks into orbits, one reaches the following double coset formula for the multiplication in  $A_+(S)$ :

$$(3.1) \quad [S/P] \cdot [S/Q] = \sum_{\bar{s} \in P \backslash S / Q} [S/(P \cap {}^s Q)],$$

where  $P \backslash S / Q$  is the set of double cosets  $PsQ$  with  $s \in S$ .

The *Burnside ring* of  $S$ , denoted  $A(S)$ , is constructed as the Grothendieck group of  $A_+(S)$ , consisting of formal differences of finite  $S$ -sets. Additively,  $A(S)$  is a free abelian group with the same basis as  $A_+(S)$ . For each element  $X \in A(S)$  we define  $c_P(X)$ , with  $P \leq S$ , to be the coefficients when we write  $X$  as a linear combination of the basis elements  $[S/P]$  in  $A(S)$ , i.e.

$$X = \sum_{[P] \in Cl(S)} c_P(X) \cdot [S/P].$$

Where  $Cl(S)$  denotes the set of  $S$ -conjugacy classes of subgroup in  $S$ .

The resulting maps  $c_P: A(S) \rightarrow \mathbb{Z}$  are group homomorphisms, but they are *not* ring homomorphisms. Note also that an element  $X$  is in  $A_+(S)$ , i.e.  $X$  is an  $S$ -set, if and only if  $c_P(X) \geq 0$  for all  $P \leq S$ .

Instead of counting orbits, an alternative way of characterising an  $S$ -set is counting the fixed points for each subgroup  $P \leq S$ . For every  $P \leq S$  and  $S$ -set  $X$ , we denote the number of fixed points by  $\Phi_P(X) := |X^P|$ , and this number only depends on  $P$  up to  $S$ -conjugation. Since we have

$$|(X \sqcup Y)^P| = |X^P| + |Y^P|, \quad \text{and} \quad |(X \times Y)^P| = |X^P| \cdot |Y^P|$$

for all  $S$ -sets  $X$  and  $Y$ , the *fixed point map*  $\Phi_P: A_+(S) \rightarrow \mathbb{Z}$  extends to a ring homomorphism  $\Phi_P: A(S) \rightarrow \mathbb{Z}$ . On the basis elements  $[S/P]$ , the number of fixed points is given by

$$\Phi_P([S/P]) = |(S/P)^P| = \frac{|N_S(Q, P)|}{|P|},$$

where  $N_S(Q, P) = \{s \in S \mid {}^s Q \leq P\}$  is the transporter in  $S$  from  $Q$  to  $P$ . In particular,  $\Phi_P([S/P]) \neq 0$  if and only if  $Q \lesssim_S P$  ( $Q$  is conjugate to a subgroup of  $P$ ).

We have one fixed point homomorphism  $\Phi_P$  per conjugacy class of subgroups in  $S$ , and we combine them into the *homomorphism of marks*  $\Phi = \Phi^S: A(S) \xrightarrow{\prod_{[P]} \Phi_P} \prod_{[P] \in Cl(S)} \mathbb{Z}$ . This ring homomorphism maps  $A(S)$  into the product ring  $\tilde{\Omega}(S) := \prod_{[P] \in Cl(S)} \mathbb{Z}$  which is the so-called *ghost ring* for the Burnside ring  $A(S)$ .

Results by Dress and others show that the mark homomorphism is injective, and that the cokernel of  $\Phi$  is the *obstruction group*  $Obs(S) := \prod_{[P] \in Cl(S)} (\mathbb{Z}/|W_S P| \mathbb{Z})$  – where  $W_S P := N_S P / P$ . These statements are combined in the following proposition, the proof of which can be found in [4], [3, Chapter 1] and [7].

**Proposition 3.1.** *Let  $\Psi = \Psi^S: \tilde{\Omega}(S) \rightarrow Obs(S)$  be given by the  $[P]$ -coordinate functions*

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|}.$$

Here  $\xi_{\langle s \rangle P}$  denotes the  $[\langle s \rangle P]$ -coordinate of an element  $\xi \in \tilde{\Omega}(S) = \prod_{[P] \in Cl(S)} \mathbb{Z}$ .

The following sequence of abelian groups is then exact:

$$0 \rightarrow A(S) \xrightarrow{\Phi} \tilde{\Omega}(S) \xrightarrow{\Psi} \text{Obs}(S) \rightarrow 0.$$

$\Phi$  is a ring homomorphism, and  $\Psi$  is just a group homomorphism.

The strength of this result is that it enables one to perform calculations for the Burnside ring  $A(S)$  inside the much nicer product ring  $\tilde{\Omega}(S)$ , where we identify each element  $X \in A(S)$  with its fixed point vector  $(\Phi_Q(X))_{[Q] \in \text{Cl}(S)}$ .

#### 4. STABLE SETS FOR A FUSION SYSTEM

Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ . In this section we rephrase the property of  $\mathcal{F}$ -stability in terms of the fixed point homomorphisms, and show in example 4.3 how theorem A can fail for a group  $G$  if we only consider  $S$ -sets that are restrictions of  $G$ -sets, instead of considering all  $G$ -stable sets. We also consider two possible definitions for the Burnside ring of a fusion system – these agree if  $\mathcal{F}$  is saturated. The proof of theorem A begins in section 4.1 in earnest.

Recall that a finite  $S$ -set  $X$  is said to be  $\mathcal{F}$ -stable if it satisfies (1.2):

$_{P,\varphi}X$  is isomorphic to  $_{P,\text{incl}}X$  as  $P$ -sets, for all  $P \leq S$  and homomorphisms  $\varphi: P \rightarrow S$  in  $\mathcal{F}$ .

In order to define  $\mathcal{F}$ -stability not just for  $S$ -sets, but for all elements of the Burnside ring, we extend  $_{P,\varphi}X$  to all  $X \in A(S)$ . Given a homomorphism  $\varphi \in \mathcal{F}(P, S)$  and an  $S$ -set  $X$ , the  $P$ -set  $_{P,\varphi}X$  was defined as  $X$  with the action restricted along  $\varphi$ , that is  $p.x := \varphi(p)x$  for  $x \in X$  and  $p \in P$ . This construction then extends linearly to a ring homomorphism  $r_\varphi: A(S) \rightarrow A(P)$ , and we denote  $_{P,\varphi}X := r_\varphi(X)$  for all  $X \in A(S)$ . In this way (1.2) makes sense for all  $X \in A(S)$ .

Additionally, it is possible to state  $\mathcal{F}$ -stability purely in terms of fixed points and the homomorphism of marks for  $A(S)$ .

**Lemma 4.1** ([5]). *The following are equivalent for all elements  $X \in A(S)$ :*

- (i)  $_{P,\varphi}X = _{P,\text{incl}}X$  in  $A(P)$  for all  $\varphi \in \mathcal{F}(P, S)$  and  $P \leq S$ .
- (ii)  $\Phi_P(X) = \Phi_{\varphi P}(X)$  for all  $\varphi \in \mathcal{F}(P, S)$  and  $P \leq S$ .
- (iii)  $\Phi_P(X) = \Phi_Q(X)$  for all pairs  $P, Q \leq S$  with  $P \sim_{\mathcal{F}} Q$ .

We shall primarily use (ii) and (iii) to characterize  $\mathcal{F}$ -stability.

*Proof.* Let  $\Phi^P: A(P) \rightarrow \tilde{\Omega}(P)$  be the homomorphism of marks for  $A(P)$ , and note that  $\Phi_R^P(_{P,\text{incl}}X) = \Phi_R(X)$  for all  $R \leq P \leq S$ .

By the definition of the  $P$ -action on  $_{P,\varphi}X$ , we have  $(_{P,\varphi}X)^R = X^{\varphi R}$  for any  $S$ -set  $X$  and all subgroups  $R \leq P$ . This generalizes to

$$\Phi_R^P(_{P,\varphi}X) = \Phi_{\varphi R}(X)$$

for  $X \in A(S)$ .

Assume (i). Then we immediately get

$$\Phi_P(X) = \Phi_P^P(_{P,\text{incl}}X) = \Phi_P^P(_{P,\varphi}X) = \Phi_{\varphi P}(X)$$

for all  $P \leq S$  and  $\varphi \in \mathcal{F}(P, S)$ ; which proves (i)  $\Rightarrow$  (ii).

Assume (ii). Let  $P \leq S$  and  $\varphi \in \mathcal{F}(P, S)$ . By assumption, we have  $\Phi_{\varphi R}(X) = \Phi_R(X)$  for all  $R \leq P$ , hence

$$\Phi_R^P(_{P,\varphi}X) = \Phi_{\varphi R}(X) = \Phi_R(X) = \Phi_R^P(_{P,\text{incl}}X).$$

Since  $\Phi^P$  is injective, we get  ${}_{P,\varphi}X = {}_{P, \text{incl}}X$ ; so (ii) $\Rightarrow$ (i).

Finally, we have (ii) $\Leftrightarrow$ (iii) because  $Q$  is  $\mathcal{F}$ -conjugate to  $P$  exactly when  $Q$  is the image of a map  $\varphi \in \mathcal{F}(P, S)$  in the fusion system.  $\square$

**Definition 4.2.** We let  $A_+(\mathcal{F}) \subseteq A_+(S)$  be the set of all the  $\mathcal{F}$ -stable sets, and by property (iii) the sums and products of stable elements are still stable, so  $A_+(\mathcal{F})$  is a subsemiring of  $A_+(S)$ .

Suppose that  $\mathcal{F} = \mathcal{F}_S(G)$  is the fusion system for a group with  $S \in \text{Syl}_p(G)$ . Let  $X \in A_+(G)$  be a  $G$ -set, and let  ${}_S X$  be the same set with the action restricted to the Sylow  $p$ -subgroup  $S$ . If we let  $P \leq S$  and  $c_g \in \text{Hom}_{\mathcal{F}_S(G)}(P, S)$  be given; then  $x \mapsto gx$  is an isomorphism of  $P$ -sets  ${}_{P, \text{incl}}X \cong {}_{P, c_g}X$ . The restriction  ${}_{S, \text{incl}}X$  is therefore  $G$ -stable.

Restricting the group action from  $G$  to  $S$  therefore defines a homomorphism of semirings  $A_+(G) \rightarrow A_+(\mathcal{F}_S(G))$ , but as the following example shows, this map need not be injective nor surjective.

**Example 4.3.** The symmetric group  $S_5$  on 5 letters has Sylow 2-subgroups isomorphic to the dihedral group  $D_8$  of order 8. We then consider  $D_8$  as embedding in  $S_5$  as one of the Sylow 2-subgroups. Let  $H, K$  be respectively Sylow 3- and 5-subgroups of  $S_5$ .

The transitive  $S_5$ -set  $[S_5/H]$  contains 40 elements and all the stabilizers have odd order (they are conjugate to  $H$ ). When we restrict the action to  $D_8$ , the stabilizers therefore become trivial so the  $D_8$ -action is free, hence  $[S_5/H]$  restricts to the  $D_8$ -set  $5 \cdot [D_8/1]$ , that is 5 disjoint copies of the free orbit  $[D_8/1]$ . Similarly, the transitive  $S_5$ -set  $[S_5/K]$  restricts to  $3 \cdot [D_8/1]$ .

These two restrictions of  $S_5$ -sets are not linearly independent as  $D_8$ -sets – the  $S_5$ -sets  $3 \cdot [S_5/H]$  and  $5 \cdot [S_5/K]$  both restrict to  $15 \cdot [D_8/1]$ . If the restrictions of  $S_5$ -sets were to form a free abelian monoid, then the set  $[D_8/1]$  would have to be the restriction of an  $S_5$ -set as well; and since  $[D_8/1]$  is irreducible as a  $D_8$ -set, it would have to be the restriction of an irreducible (hence transitive)  $S_5$ -set. However  $S_5$  has no subgroup of index 8, hence there is no transitive  $S_5$  with 8 elements.

This shows that the restrictions of  $S_5$ -sets to  $D_8$  do not form a free abelian monoid, and we also see that  $[D_8/1]$  is an example of an  $\mathcal{F}_{D_8}(S_5)$ -stable set ( $\Phi_1([D_8/1]) = 8$  and  $\Phi_Q([D_8/1]) = 0$  for  $1 \neq Q \leq D_8$ ) which cannot be given the structure of an  $S_5$ -set.

To define the Burnside ring of a fusion system  $\mathcal{F}$ , we have two possibilities: We can consider the semiring of all the  $\mathcal{F}$ -stable  $S$ -sets and take the Grothendieck group of this. Alternatively, we can first take the Grothendieck group for all  $S$ -sets to get the Burnside ring of  $S$ , and then afterwards we consider the subring herein consisting of all the  $\mathcal{F}$ -stable elements. The following proposition implies that the two definitions coincide for saturated fusion systems.

**Proposition 4.4.** *Let  $\mathcal{F}$  be a fusion system over a  $p$ -group  $S$ , and consider the subsemiring  $A_+(\mathcal{F})$  of  $\mathcal{F}$ -stable  $S$ -sets in the semiring  $A_+(S)$  of finite  $S$ -sets.*

*This inclusion induces a ring homomorphism from the Grothendieck group of  $A_+(\mathcal{F})$  to the Burnside ring  $A(S)$ , which is injective.*

*If  $\mathcal{F}$  is saturated, then the image of the homomorphism is the subring of  $A(S)$  consisting of the  $\mathcal{F}$ -stable elements.*

*Proof.* Let  $Gr$  be the Grothendieck group of  $A_+(\mathcal{F})$ , and let  $I: Gr \rightarrow A(S)$  be the induced map coming from the inclusion  $i: A_+(\mathcal{F}) \hookrightarrow A_+(S)$ .

An element of  $Gr$  is a formal difference  $X - Y$  where  $X$  and  $Y$  are  $\mathcal{F}$ -stable sets. Assume now that  $X - Y$  lies in  $\ker I$ . This means that  $i(X) - i(Y) = 0$  in  $A(S)$ ; and since  $A_+(S)$

is a free commutative monoid, we conclude that  $i(X) = i(Y)$  as  $S$ -sets. But  $i$  is just the inclusion map, so we must have  $X = Y$  in  $A_+(\mathcal{F})$  as well, and  $X - Y = 0$  in  $Gr$ . Hence  $I: Gr \rightarrow A(S)$  is injective.

It is clear that the difference of two  $\mathcal{F}$ -stable sets is still  $\mathcal{F}$ -stable, so  $\text{im } I$  lies in the subring of  $\mathcal{F}$ -stable elements. If  $\mathcal{F}$  is saturated, then the converse holds, and all  $\mathcal{F}$ -stable elements of  $A(S)$  can be written as a difference of  $\mathcal{F}$ -stable sets; however the proof of this must be postponed to corollary 4.11 below.  $\square$

**Definition 4.5.** Let  $\mathcal{F}$  be saturated. We define the *Burnside ring of  $\mathcal{F}$* , denoted  $A(\mathcal{F})$ , to be the subring consisting of the  $\mathcal{F}$ -stable elements in  $A(S)$ .

Once we have proven corollary 4.11, we will know that  $A(\mathcal{F})$  is also the Grothendieck group of the semiring  $A_+(\mathcal{F})$  of  $\mathcal{F}$ -stable sets.

**4.1. Proving theorems A and B.** The proof of theorem A falls into several parts: We begin by constructing some  $\mathcal{F}$ -stable sets  $\alpha_P$  satisfying certain properties – this is the content of 4.6-4.8. We construct one  $\alpha_P$  per  $\mathcal{F}$ -conjugacy class of subgroups, and these are the  $\mathcal{F}$ -stable sets which we will later show are the irreducible stable sets. A special case of the construction was originally used by Broto, Levi and Oliver in [2, Proposition 5.5] to show that every saturated fusion system has a characteristic biset.

In 4.9-4.11 we then proceed to show that the constructed  $\alpha_P$ 's are linearly independent, and that they generate the Burnside ring  $A(\mathcal{F})$ . When proving that the  $\alpha_P$ 's generate  $A(\mathcal{F})$ , the same proof also establishes theorem B.

Finally, we use the fact that the  $\alpha_P$ 's form a basis for the Burnside ring, to argue that they form an additive basis already for the semiring  $A_+(\mathcal{F})$ , completing the proof of theorem A itself.

As mentioned, we first construct an  $\mathcal{F}$ -stable set  $\alpha_P$  for each  $\mathcal{F}$ -conjugation class of subgroups. The idea when constructing  $\alpha_P$  is that we start with the single orbit  $[S/P]$  which we then stabilize: We run through the subgroups  $Q \leq S$  in decreasing order and add orbits to the constructed  $S$ -set such that it becomes  $\mathcal{F}$ -stable at the conjugacy class of  $Q$  in  $\mathcal{F}$ . The stabilization procedure is handled in the following technical lemma 4.6, which is then applied in proposition 4.8 to construct the  $\alpha_P$ 's.

Recall that  $c_P(X)$  denotes the number of  $(S/P)$ -orbits in  $X$ , and  $\Phi_P(X)$  denotes the number of  $P$ -fixed points.

**Lemma 4.6.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ , and let  $\mathcal{H}$  be a collection of subgroups of  $S$  such that  $\mathcal{H}$  is closed under taking  $\mathcal{F}$ -subconjugates, i.e. if  $P \in \mathcal{H}$ , then  $Q \in \mathcal{H}$  for all  $Q \lesssim_{\mathcal{F}} P$ .*

*Assume that  $X \in A_+(S)$  is an  $S$ -set satisfying  $\Phi_P(X) = \Phi_{P'}(X)$  for all pairs  $P \sim_{\mathcal{F}} P'$ , with  $P, P' \notin \mathcal{H}$ . Assume furthermore that  $c_P(X) = 0$  for all  $P \in \mathcal{H}$ .*

*Then there exists an  $\mathcal{F}$ -stable set  $X' \in A_+(\mathcal{F}) \subseteq A_+(S)$  satisfying  $\Phi_P(X') = \Phi_P(X)$  and  $c_P(X') = c_P(X)$  for all  $P \notin \mathcal{H}$ ; and also satisfying  $c_P(X') = c_P(X)$  for all  $P \leq S$  which are fully normalized in  $\mathcal{F}$ . In particular, for a  $P \in \mathcal{H}$  which is fully normalized, we have  $c_P(X') = 0$ .*

*Proof.* We proceed by induction on the size of the collection  $\mathcal{H}$ . If  $\mathcal{H} = \emptyset$ , then  $X$  is  $\mathcal{F}$ -stable by assumption, so  $X' := X$  works.

Assume that  $\mathcal{H} \neq \emptyset$ , and let  $P \in \mathcal{H}$  be maximal under  $\mathcal{F}$ -subconjugation as well as fully normalized.



Let  $P' \sim_{\mathcal{F}} P$ . Then there is a homomorphism  $\varphi \in \mathcal{F}(N_S P', N_S P)$  with  $\varphi(P') = P$  by lemma 2.3 since  $\mathcal{F}$  is saturated. The restriction of  $S$ -actions to  $\varphi(N_S P')$  gives a ring homomorphism  $A(S) \rightarrow A(\varphi(N_S P'))$  that preserves the fixed-point homomorphisms  $\Phi_Q$  for  $Q \leq \varphi(N_S P') \leq N_S P$ .

If we consider the  $S$ -set  $X$  as an element of  $A(\varphi(N_S P'))$ , we can apply the short exact sequence of proposition 3.1 to get  $\Psi^{\varphi(N_S P')}(\Phi^{\varphi(N_S P')}(X)) = 0$ . In particular, the  $P$ -coordinate function satisfies  $\Psi_P^{\varphi(N_S P')}(\Phi^{\varphi(N_S P')}(X)) = 0$ , that is

$$\sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) \equiv 0 \pmod{|\varphi(N_S P')/P|}.$$

Similarly, we have  $\Psi^S(\Phi^S(X)) = 0$ , where the  $P'$ -coordinate  $\Psi_{P'}^S(\Phi^S(X)) = 0$  gives us

$$\sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \equiv 0 \pmod{|N_S P'/P'|}.$$

Since  $P$  is maximal in  $\mathcal{H}$ , we have by assumption  $\Phi_Q(X) = \Phi_{Q'}(X)$  for all  $Q \sim_{\mathcal{F}} Q'$  where  $P$  is  $\mathcal{F}$ -conjugate to a *proper* subgroup of  $Q$ . Specifically, we have

$$\Phi_{\langle \varphi(s) \rangle P}(X) = \Phi_{\varphi(\langle s \rangle P')}(X) = \Phi_{\langle s \rangle P'}(X)$$

for all  $s \in N_S P'$  with  $s \notin P'$ . It then follows that

$$\begin{aligned} \Phi_P(X) - \Phi_{P'}(X) &= \sum_{\bar{s} \in \varphi(N_S P')/P} \Phi_{\langle s \rangle P}(X) - \sum_{\bar{s} \in N_S P'/P'} \Phi_{\langle s \rangle P'}(X) \\ &\equiv 0 - 0 \pmod{|W_S P'|}. \end{aligned}$$

We can therefore define  $\lambda_{P'} := (\Phi_P(X) - \Phi_{P'}(X))/|W_S P'| \in \mathbb{Z}$ .

Using the  $\lambda_{P'}$  as coefficients, we construct a new  $S$ -set

$$\tilde{X} := X + \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \lambda_{P'} \cdot [S/P'] \in A(S).$$

Here  $[P]_{\mathcal{F}}$  is the collection of subgroups that are  $\mathcal{F}$ -conjugate to  $P$ . The sum is then taken over one representative from each  $S$ -conjugacy class contained in  $[P]_{\mathcal{F}}$ .

A priori, the  $\lambda_{P'}$  might be negative, and as a result  $\tilde{X}$  might not be an  $S$ -set. In the original construction of [2], this problem is circumvented by adding copies of

$$\sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} \cdot [S/P']$$

until all the coefficients are non-negative.

It will however be shown in lemma 4.7 below, that under the assumption that  $c_{P'}(X) = 0$  for  $P' \sim_{\mathcal{F}} P$ , then  $\lambda_{P'}$  is always non-negative, and  $\lambda_{P'} = 0$  if  $P'$  is fully normalized. Hence  $\tilde{X}$  is already an  $S$ -set without further adjustments.

We clearly have  $c_Q(\tilde{X}) = c_Q(X)$  for all  $Q \not\sim_{\mathcal{F}} P$ , in particular for all  $Q \notin \mathcal{H}$ . Furthermore, if  $P' \sim_{\mathcal{F}} P$  is fully normalized, then  $c_{P'}(\tilde{X}) = c_{P'}(X) + \lambda_{P'} = c_{P'}(X)$ .

Because  $\Phi_Q([S/P']) = 0$  unless  $Q \lesssim_S P'$ , we see that  $\Phi_Q(\tilde{X}) = \Phi_Q(X)$  for every  $Q \notin \mathcal{H}$ . Secondly, we calculate  $\Phi_{P'}(\tilde{X})$  for each  $P' \sim_{\mathcal{F}} P$ :

$$\begin{aligned} \Phi_{P'}(\tilde{X}) &= \Phi_{P'}(X) + \sum_{[\tilde{P}]_S \subseteq [P]_{\mathcal{F}}} \lambda_{\tilde{P}} \cdot \Phi_{P'}([S/\tilde{P}]) \\ &= \Phi_{P'}(X) + \lambda_{P'} \cdot \Phi_{P'}([S/P']) = \Phi_{P'}(X) + \lambda_{P'} |W_S P'| \\ &= \Phi_P(X); \end{aligned}$$

which is independent of the choice of  $P' \sim_{\mathcal{F}} P$ .

We define  $\mathcal{H}' := \mathcal{H} \setminus [P]_{\mathcal{F}}$  as  $\mathcal{H}$  with the  $\mathcal{F}$ -conjugates of  $P$  removed. Because  $P$  is maximal in  $\mathcal{H}$ , the subcollection  $\mathcal{H}'$  again contains all  $\mathcal{F}$ -subconjugates of any  $H \in \mathcal{H}'$ .

By induction we can apply lemma 4.6 to  $\tilde{X}$  and the smaller collection  $\mathcal{H}'$ . We get an  $X' \in A_+(\mathcal{F})$  with  $\Phi_Q(X') = \Phi_Q(\tilde{X})$  and  $c_Q(X') = c_Q(\tilde{X})$  for all  $Q \notin \mathcal{H}'$ ; and such that  $c_Q(X') = 0$  if  $Q \in \mathcal{H}'$  is fully normalized.

It follows that  $\Phi_Q(X') = \Phi_Q(\tilde{X}) = \Phi_Q(X)$  and  $c_Q(X') = c_Q(\tilde{X}) = c_Q(X)$  for all  $Q \notin \mathcal{H}$ , and we also have  $c_Q(X') = 0$  if  $Q \in \mathcal{H}$  is fully normalized.  $\square$

**Lemma 4.7.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ , and let  $P \leq S$  be a fully normalized subgroup.*

*Suppose that  $X$  is an  $S$ -set with  $c_{P'}(X) = 0$  for all  $P' \sim_{\mathcal{F}} P$ , and satisfying that  $X$  is already  $\mathcal{F}$ -stable for subgroups larger than  $P$ , i.e.  $|X^R| = |X^{R'}|$  for all  $R \sim_{\mathcal{F}} R'$  where  $P$  is  $\mathcal{F}$ -conjugate to a proper subgroup of  $R$ .*

*Then  $|X^P| \geq |X^{P'}|$  for all  $P' \sim_{\mathcal{F}} P$ .*

*Proof.* Let  $Q \sim_{\mathcal{F}} P$  be given. Because  $P$  is fully normalized, there exists by lemma 2.3 a homomorphism  $\varphi: N_S Q \hookrightarrow N_S P$  in  $\mathcal{F}$ , with  $\varphi(Q) = P$ .

Let  $A_1, \dots, A_k$  be the subgroups of  $N_S Q$  that strictly contain  $Q$ , i.e.  $Q < A_i \leq N_S Q$ . We put  $B_i := \varphi(A_i)$ , and thus also have  $P < B_i \leq N_S P$ . We let  $C_1, \dots, C_\ell$  be the subgroups of  $N_S P$  strictly containing  $P$  which are not the image (under  $\varphi$ ) of some  $A_i$ . Hence  $B_1, \dots, B_k, C_1, \dots, C_\ell$  are all the different subgroups of  $N_S P$  strictly containing  $P$ . We denote the set  $\{1, \dots, k\}$  of indices by  $I$ , and also  $J := \{1, \dots, \ell\}$ .

Because  $c_Q(X) = c_P(X) = 0$  by assumption, no orbit of  $X$  is isomorphic to  $S/Q$ , hence no element in  $X^Q$  has  $Q$  as a stabilizer. Let  $x \in X^Q$  be any element, and let  $K > Q$  be the stabilizer of  $x$ ; so  $x \in X^K \subseteq X^Q$ . Since  $K$  is a  $p$ -group, there is some intermediate group  $L$  with  $Q \triangleleft L \leq K$ ; hence  $x \in X^L$  for some  $Q < L \leq N_S Q$ . We conclude that

$$X^Q = \bigcup_{i \in I} X^{A_i}.$$

With similar reasoning we also get

$$X^P = \bigcup_{i \in I} X^{B_i} \cup \bigcup_{j \in J} X^{C_j}.$$

The proof is then completed by showing

$$|X^P| = \left| \bigcup_{i \in I} X^{B_i} \cup \bigcup_{j \in J} X^{C_j} \right| \geq \left| \bigcup_{i \in I} X^{B_i} \right| \stackrel{(*)}{=} \left| \bigcup_{i \in I} X^{A_i} \right| = |X^Q|.$$

We only need to prove the equality  $(*)$ .

Showing  $(*)$  has only to do with fixed points for the subgroups  $A_i$  and  $B_i$ ; and because  $B_i = \varphi(A_i) \sim_{\mathcal{F}} A_i$  are subgroups that strictly contain  $P$  and  $Q$  respectively, we have  $|X^{B_i}| = |X^{A_i}|$  by assumption.

To get  $(*)$  for the unions  $\cup A_i$  and  $\cup B_i$  we then have to apply the inclusion-exclusion principle:

$$\left| \bigcup_{i \in I} X^{B_i} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| \bigcap_{i \in \Lambda} X^{B_i} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| X^{\langle B_i \rangle_{i \in \Lambda}} \right|.$$

Here  $\langle B_i \rangle_{i \in \Lambda} \leq N_S P$  is the subgroup generated by the elements of  $B_i$ 's with  $i \in \Lambda \subseteq I$ . Recalling that  $B_i = \varphi(A_i)$  by definition, we have  $\langle B_i \rangle_{i \in \Lambda} = \langle \varphi(A_i) \rangle_{i \in \Lambda} = \varphi(\langle A_i \rangle_{i \in \Lambda})$ , and consequently

$$\sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| X^{\langle B_i \rangle_{i \in \Lambda}} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| X^{\varphi(\langle A_i \rangle_{i \in \Lambda})} \right|.$$

Because  $Q < A_i \leq N_S Q$ , we also have  $Q < \langle A_i \rangle_{i \in \Lambda} \leq N_S Q$ , by assumption we therefore get  $\left| X^{\varphi(\langle A_i \rangle_{i \in \Lambda})} \right| = \left| X^{\langle A_i \rangle_{i \in \Lambda}} \right|$  for all  $\emptyset \neq \Lambda \subseteq I$ . It then follows that

$$\sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| X^{\varphi(\langle A_i \rangle_{i \in \Lambda})} \right| = \sum_{\emptyset \neq \Lambda \subseteq I} (-1)^{|\Lambda|+1} \left| X^{\langle A_i \rangle_{i \in \Lambda}} \right| = \dots = \left| \bigcup_{i \in I} X^{A_i} \right|,$$

where we use the inclusion-exclusion principle in reverse. We have thus shown the equality  $\left| \bigcup_{i \in I} X^{B_i} \right| = \left| \bigcup_{i \in I} X^{A_i} \right|$  as required.  $\square$

Applying the technical lemma 4.6, we can now construct the irreducible  $\mathcal{F}$ -stable sets  $\alpha_P$  for  $P \leq S$  as described in the following proposition. That the  $\alpha_P$ 's are in fact irreducible, or even that they are unique, will not be shown until the proof of theorem A itself.

**Proposition 4.8.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ .*

*For each  $\mathcal{F}$ -conjugacy class  $[P]_{\mathcal{F}} \in Cl(\mathcal{F})$  of subgroups, there is an  $\mathcal{F}$ -stable set  $\alpha_P \in A_+(\mathcal{F})$  such that*

- (i)  $\Phi_Q(\alpha_P) = 0$  unless  $Q$  is  $\mathcal{F}$ -subconjugate to  $P$ .
- (ii)  $c_{P'}(\alpha_P) = 1$  and  $\Phi_{P'}(\alpha_P) = |W_S P'|$  when  $P'$  is fully normalized and  $P' \sim_{\mathcal{F}} P$ .
- (iii)  $c_Q(\alpha_P) = 0$  when  $Q$  is fully normalized and  $Q \not\sim_{\mathcal{F}} P$ .

*Proof.* Let  $P \leq S$  be fully  $\mathcal{F}$ -normalized. We let  $X \in A_+(S)$  be the  $S$ -set

$$X := \sum_{[P']_S \subseteq [P]_{\mathcal{F}}} \frac{|N_S P|}{|N_S P'|} \cdot [S/P'] \in A_+(S).$$

$X$  then satisfies that  $\Phi_Q(X) = 0$  unless  $Q \lesssim_S P'$  for some  $P' \sim_{\mathcal{F}} P$ , in which case we have  $Q \lesssim_{\mathcal{F}} P$ . For all  $P', P'' \in [P]_{\mathcal{F}}$  we have  $\Phi_{P''}([S/P']) = 0$  unless  $P'' \sim_S P'$ ; and consequently

$$\Phi_{P'}(X) = \frac{|N_S P|}{|N_S P'|} \cdot \Phi_{P'}([S/P']) = \frac{|N_S P|}{|N_S P'|} \cdot |W_S P'| = |W_S P|$$

which doesn't depend on  $P' \sim_{\mathcal{F}} P$ .

Let  $\mathcal{H}$  be the collection of all  $Q$  which are  $\mathcal{F}$ -conjugate to a proper subgroup of  $P$ , then  $\Phi_Q(X) = \Phi_{Q'}(X)$  for all pairs  $Q \sim_{\mathcal{F}} Q'$  not in  $\mathcal{H}$ . Using lemma 4.6 we get some  $\alpha_P \in A_+(\mathcal{F})$  with the required properties.  $\square$

Properties (ii) and (iii) make it really simple to decompose a linear combination  $X$  of the  $\alpha_P$ 's. The coefficient of  $\alpha_P$  in  $X$  is just the number of  $[S/P]$ -orbits in  $X$  as an  $S$ -set - when  $P$  is fully normalized. This is immediate since  $\alpha_P$  contains exactly one copy of  $[S/P]$ , and no other  $\alpha_Q$  contains  $[S/P]$ .

In particular we have:

**Corollary 4.9.** *The  $\alpha_P$ 's in proposition 4.8 are linearly independent.*

In order to prove that the  $\alpha_P$ 's generate all  $\mathcal{F}$ -stable sets, we will first show that the  $\alpha_P$ 's generate all the  $\mathcal{F}$ -stable elements in the Burnside ring. As a tool for proving this, we define a ghost ring for the Burnside ring  $A(\mathcal{F})$ ; and as consequence of how the proof proceeds, we end up showing an analogue of proposition 3.1 for saturated fusion systems, describing how the Burnside ring  $A(\mathcal{F})$  lies embedded in the ghost ring – this is the content of theorem B.

**Definition 4.10.** Recall how the ghost ring  $\tilde{\Omega}(S)$  for the Burnside ring of a group is defined as the product ring  $\prod_{[P]_S \in Cl(S)} \mathbb{Z}$  where the coordinates correspond to the  $S$ -conjugacy classes of subgroups. For the ring  $A(\mathcal{F})$ , we now similarly define *the ghost ring*  $\tilde{\Omega}(\mathcal{F})$  as a product ring  $\prod_{[P]_{\mathcal{F}} \in Cl(\mathcal{F})} \mathbb{Z}$  with coordinates corresponding to the  $\mathcal{F}$ -conjugacy classes of subgroups.

The surjection of indexing sets  $Cl(S) \rightarrow Cl(\mathcal{F})$  which sends an  $S$ -conjugacy class  $[P]_S$  to its  $\mathcal{F}$ -conjugacy class  $[P]_{\mathcal{F}}$ , induces a homomorphism  $\tilde{\Omega}(\mathcal{F}) \hookrightarrow \tilde{\Omega}(S)$  that embeds  $\tilde{\Omega}(\mathcal{F})$  as the subring of vectors which are constant on each  $\mathcal{F}$ -conjugacy class.

Since  $A(\mathcal{F})$  is the subring of  $\mathcal{F}$ -stable elements in  $A(S)$ , we can restrict the mark homomorphism  $\Phi^S: A(S) \rightarrow \tilde{\Omega}(S)$  to the subring  $A(\mathcal{F})$  and get an injective ring homomorphism  $\Phi^{\mathcal{F}}: A(\mathcal{F}) \rightarrow \tilde{\Omega}(\mathcal{F})$  – which is the *homomorphism of marks* for  $A(\mathcal{F})$ .

To model the cokernel of  $\Phi^{\mathcal{F}}$  we define  $Obs(\mathcal{F})$  as

$$Obs(\mathcal{F}) := \prod_{\substack{[P] \in Cl(\mathcal{F}) \\ P \text{ f.n.}}} (\mathbb{Z}/|W_S P|\mathbb{Z}),$$

where 'f.n.' is short for 'fully normalized', so we take fully normalized representatives of the conjugacy classes in  $\mathcal{F}$ .

**Theorem B.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ , and let  $A(\mathcal{F})$  be the Burnside ring of  $\mathcal{F}$  – i.e. the subring consisting of the  $\mathcal{F}$ -stable elements in the Burnside ring of  $S$ .*

*We then have a short-exact sequence*

$$0 \rightarrow A(\mathcal{F}) \xrightarrow{\Phi} \tilde{\Omega}(\mathcal{F}) \xrightarrow{\Psi} Obs(\mathcal{F}) \rightarrow 0.$$

where  $\Phi = \Phi^{\mathcal{F}}$  is the homomorphism of marks, and  $\Psi = \Psi^{\mathcal{F}}: \tilde{\Omega}(\mathcal{F}) \rightarrow Obs(\mathcal{F})$  is a group homomorphism given by the  $[P]$ -coordinate functions

$$\Psi_P(\xi) := \sum_{\bar{s} \in W_S P} \xi_{\langle s \rangle P} \pmod{|W_S P|}$$

when  $P$  is a fully normalized representative of the conjugacy class  $[P]$  in  $\mathcal{F}$ . Here  $\Psi_P = \Psi_{P'}$  if  $P \sim_{\mathcal{F}} P'$  are both fully normalized.

*Proof.* We choose some total order of the conjugacy classes  $[P], [Q] \in Cl(\mathcal{F})$  such that  $|P| > |Q|$  implies  $[P] < [Q]$ , i.e. we take the subgroups in decreasing order. It holds in particular that  $Q \lesssim_{\mathcal{F}} P$  implies  $[P] \leq [Q]$ .

With respect to the ordering above, the group homomorphism  $\Psi$  is given by a lower triangular matrix with 1's in the diagonal, hence  $\Psi$  is surjective. The mark homomorphism  $\Phi = \Phi^{\mathcal{F}}$  is the restriction of the injective ring homomorphism  $\Phi^S: A(S) \rightarrow \tilde{\Omega}(S)$ , so  $\Phi$  is injective.

We know from the group case, proposition 3.1, that  $\Psi^S \circ \Phi^S = 0$ . By construction we have  $(\Psi)_P = (\Psi^S)_P$  for the coordinate functions when  $P$  is fully normalized; and  $\Phi$  is the

restriction of  $\Phi^S$ . We conclude that  $\Psi \circ \Phi = 0$  as well. It remains to be shown that  $\text{im } \Phi$  is actually all of  $\ker \Psi$ .

Consider the subgroup  $H := \text{Span}\{\alpha_P \mid [P] \in Cl(\mathcal{F})\}$  spanned by the  $\alpha_P$ 's in  $A(\mathcal{F})$ , and consider also the restriction  $\Phi|_H$  of the mark homomorphism  $\Phi: A(\mathcal{F}) \rightarrow \tilde{\Omega}(\mathcal{F})$ .

$\Phi|_H$  is described by a square matrix  $M$  in terms of the ordered bases of  $H = \text{Span}\{\alpha_P\text{'s}\}$  and  $\tilde{\Omega}(\mathcal{F})$ . Because  $M_{[Q],[P]} := \Phi_Q(\alpha_P)$  is zero unless  $P \sim_{\mathcal{F}} Q$  or  $|P| > |Q|$ , we conclude that  $M$  is a lower triangular matrix. The diagonal entries of  $M$  are

$$M_{[P],[P]} = \Phi_P(\alpha_P) = |W_S P|,$$

when  $P$  is fully normalized.

All the diagonal entries are non-zero, so the cokernel of  $\Phi|_H$  is finite of order

$$|\text{coker } \Phi|_H| = \prod_{[P] \in Cl(\mathcal{F})} M_{[P],[P]} = \prod_{\substack{[P] \in Cl(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P|.$$

Since  $\Phi|_H$  is a restriction of  $\Phi$ , it follows that  $|\text{coker } \Phi| \leq |\text{coker } \Phi|_H|$ . At the same time,  $\Psi \circ \Phi = 0$  implies that  $|\text{coker } \Phi| \geq |\text{Obs}(\mathcal{F})|$ .

We do however have

$$|\text{Obs}(\mathcal{F})| = \prod_{\substack{[P] \in Cl(\mathcal{F}) \\ P \text{ f.n.}}} |W_S P| = |\text{coker } \Phi|_H|.$$

The only possibility is that  $\ker \Psi = \text{im } \Phi = \text{im } \Phi|_H$ , completing the proof of theorem B.  $\square$

From the last equality  $\text{im } \Phi = \text{im } \Phi|_H$  and the fact that  $\Phi$  is injective, it also follows that  $A(\mathcal{F}) = H$  so the  $\alpha_P$ 's span all of  $A(\mathcal{F})$ . Combining this with corollary 4.9 we get:

**Corollary 4.11.** *The  $\alpha_P$ 's form an additive basis for the Burnside ring  $A(\mathcal{F})$ .*

The corollary tells us that any element  $X \in A(\mathcal{F})$  can be written uniquely as an integral linear combination of the  $\alpha_P$ 's. In particular, any  $\mathcal{F}$ -stable set can be written as a linear combination of  $\alpha_P$ 's, and if the coefficients are all non-negative, then we have a linear combination in  $A_+(\mathcal{F})$ .

**Theorem A.** *Let  $\mathcal{F}$  be a saturated fusion system over a  $p$ -group  $S$ .*

*The sets  $\alpha_P$  in proposition 4.8 are all the irreducible  $\mathcal{F}$ -stable sets, and every  $\mathcal{F}$ -stable set splits uniquely (up to  $S$ -isomorphism) as a disjoint union of the  $\alpha_P$ 's.*

*Hence the semiring  $A_+(\mathcal{F})$  of  $\mathcal{F}$ -stable sets is additively a free commutative monoid with rank equal to the number of conjugacy classes of subgroups in  $\mathcal{F}$ .*

*Proof.* Let  $\alpha_P \in A_+(\mathcal{F})$  for each conjugacy class  $[P] \in Cl(\mathcal{F})$  be given as in proposition 4.8. Let  $X \in A_+(\mathcal{F})$  be any  $\mathcal{F}$ -stable  $S$ -set.

Since the  $\alpha_P$ 's form a basis for  $A(\mathcal{F})$  by corollary 4.11, we can write  $X$  uniquely as

$$X = \sum_{[P] \in Cl(\mathcal{F})} \lambda_P \cdot \alpha_P$$

with  $\lambda_P \in \mathbb{Z}$ .

Suppose that  $P$  is fully normalized, then  $c_P(\alpha_Q) = 1$  if  $P \sim_{\mathcal{F}} Q$ , and  $c_P(\alpha_Q) = 0$  otherwise. As a consequence of this, we have

$$c_P(X) = \sum_{[Q] \in Cl(\mathcal{F})} \lambda_Q \cdot c_P(\alpha_Q) = \lambda_P$$

whenever  $P$  is fully normalized.

Because  $X$  is an  $S$ -set, we see that  $\lambda_P = c_P(X) \geq 0$ . Hence the linear combination  $X = \sum_{[P] \in Cl(\mathcal{F})} \lambda_P \cdot \alpha_P$  has nonnegative coefficients, i.e. it is a linear combination in the semiring  $A_+(\mathcal{F})$ .

As a special case, if we have another element  $\alpha'_P$  in  $A(\mathcal{F})$  satisfying the properties of proposition 4.8, then the fact that  $\lambda_Q = c_Q(\alpha'_P)$  for all fully normalized  $Q \leq S$ , shows that  $\lambda_P = 1$  and  $\lambda_Q = 0$  for  $Q \not\sim_{\mathcal{F}} P$ . Thus the linear combination above simplifies to  $\alpha'_P = \alpha_P$ . Hence the  $\alpha_P$ 's are uniquely determined by the properties of proposition 4.8.  $\square$

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